Transmittance Thru a Layer

Outgoing Radiance

Layer Front Transmittance

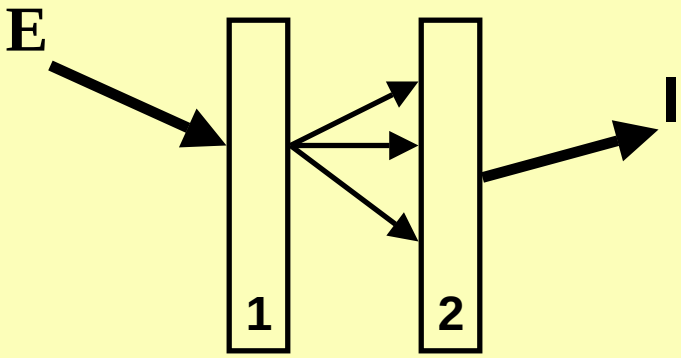
Incident Irradiance

$$I(\theta_i, \phi_i) = \tau_i^f(\underline{\theta_i, \phi_i}; \underline{\theta_{i-1}, \phi_{i-1}})E(\theta_{i-1}, \phi_{i-1})$$

Outgoing Direction

Incoming Direction

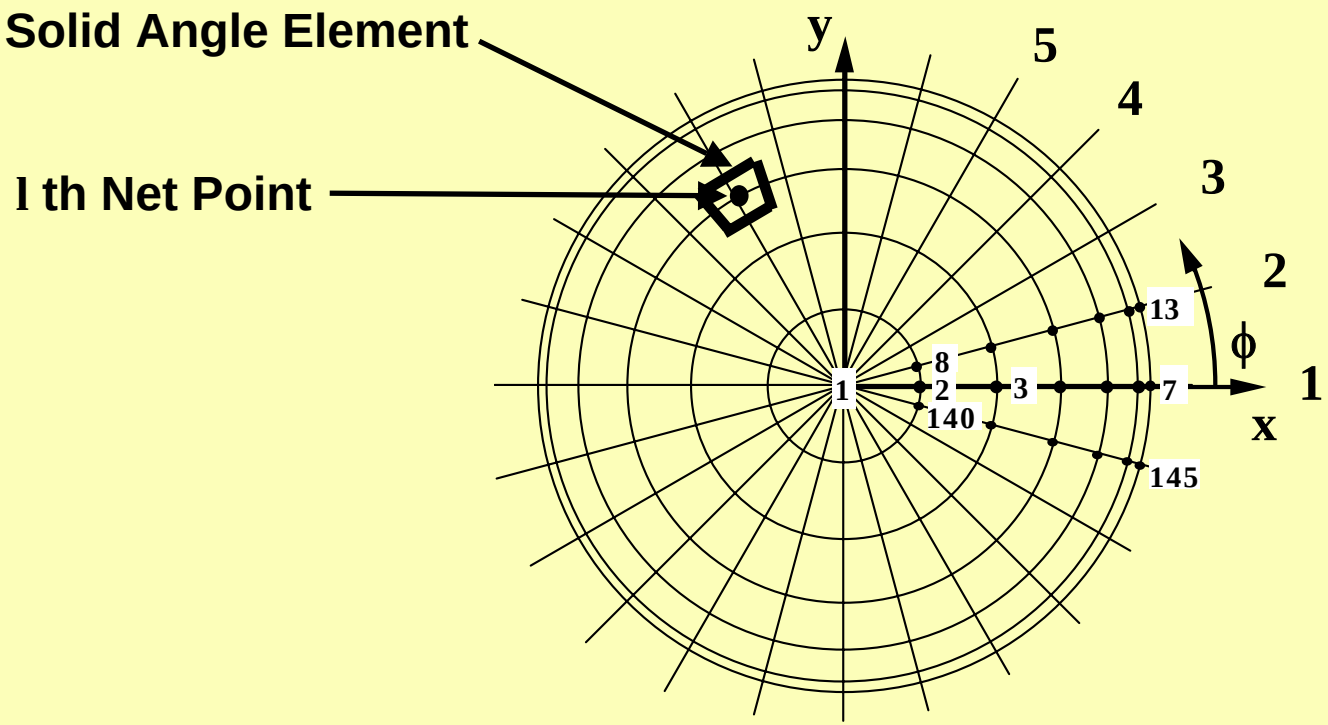
Transmission Thru a Pair of Layers



$$I(\theta_2, \phi_2) = \int d\Omega_1 \cos(\theta_1) \tau_2^f(\theta_2, \phi_2; \theta_1, \phi_1) \tau_1^f(\theta_1, \phi_1; \theta_0, \phi_0) E(\theta_0, \phi_0)$$

$$T^f_{2,\{1,2\}}(\theta_2, \phi_2; \theta_0, \phi_0) = \int d\Omega_1 \cos(\theta_1) \tau_2^f(\theta_2, \phi_2; \theta_1, \phi_1) \tau_1^f(\theta_1, \phi_1; \theta_0, \phi_0)$$

Finite-Element Partition of Solid Angle



Approximation to Integral

- **The finite-sum representation of the integral**

$$T^{f}_{2,\{1,2\}}(\theta^{m}_{2}, \phi^{m}_{2}; \theta^{k}_{0}, \phi^{k}_{0}) = \sum_{l} \tau^{f}_{2}(\theta^{m}_{2}, \phi^{m}_{2}; \theta^{l}_{1}, \phi^{l}_{1}) \Delta\Omega^{l}_{1} \cos(\theta^{l}_{1}) \tau^{f}_{1}(\theta^{l}_{1}, \phi^{l}_{1}; \theta^{k}_{0}, \phi^{k}_{0})$$

- **is very suggestive of a matrix multiplication:**

$$\left(T^{f}_{2,\{1,2\}}\right)_{mk} = \sum_{l} \left(\tau^{f}_{2}\right)_{ml} \Delta\Omega^{l}_{1} \cos(\theta^{l}_{1}) \left(\tau^{f}_{1}\right)_{lk}$$

Column-vector Representation of Functions

Irradiance

$$\mathbf{E}_{i-1} = \begin{Bmatrix} E(\theta_{i-1}^{1}, \phi_{i-1}^{1}) \\ E(\theta_{i-1}^{2}, \phi_{i-1}^{2}) \\ \dots \\ E(\theta_{i-1}^{N}, \phi_{i-1}^{N}) \end{Bmatrix}$$

Radiance

$$\mathbf{I}_{i} = \begin{Bmatrix} I(\theta_{i}^{1}, \phi_{i}^{1}) \\ I(\theta_{i}^{2}, \phi_{i}^{2}) \\ \dots \\ I(\theta_{i}^{N}, \phi_{i}^{N}) \end{Bmatrix}$$

Layer Transmittance Matrix

$$\tau_i^f = \begin{Bmatrix} \tau_i^f(\theta_i^1, \phi_i^1; \theta_{i-1}^1, \phi_{i-1}^1) & \tau_i^f(\theta_i^1, \phi_i^1; \theta_{i-1}^2, \phi_{i-1}^2) & \dots & \tau_i^f(\theta_i^1, \phi_i^1; \theta_{i-1}^N, \phi_{i-1}^N) \\ \tau_i^f(\theta_i^2, \phi_i^2; \theta_{i-1}^1, \phi_{i-1}^1) & \tau_i^f(\theta_i^2, \phi_i^2; \theta_{i-1}^2, \phi_{i-1}^2) & \dots & \dots \\ \dots & \dots & \dots & \dots \\ \tau_i^f(\theta_i^N, \phi_i^N; \theta_{i-1}^1, \phi_{i-1}^1) & \dots & \dots & \tau_i^f(\theta_i^N, \phi_i^N; \theta_{i-1}^N, \phi_{i-1}^N) \end{Bmatrix}$$

Functional Equations Replaced By Matrix Equations

$$I(\theta_i, \phi_i) = \tau_i^f(\theta_i, \phi_i; \theta_{i-1}, \phi_{i-1}) E(\theta_{i-1}, \phi_{i-1})$$

$$\downarrow$$

$$\mathbf{I}_i = \boldsymbol{\tau}_i^f \cdot \mathbf{E}_{i-1}$$

Specular Transmittance

$$\tau_i = \begin{pmatrix} \frac{\tau^{(1)}_{(S)i}}{\Lambda^{(1)}} & 0 & \dots & 0 \\ 0 & \frac{\tau^{(2)}_{(S)i}}{\Lambda^{(2)}} & 0 & \dots \\ \dots & 0 & \dots & 0 \\ 0 & \dots & 0 & \frac{\tau^{(N)}_{(S)i}}{\Lambda^{(N)}} \end{pmatrix}$$

Propagation Matrix

$$\mathbf{\Lambda}_i = \begin{Bmatrix} \Delta\Omega_i^1 \cos(\theta_i^1) & 0 & \dots & 0 \\ 0 & \Delta\Omega_i^2 \cos(\theta_i^2) & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \Delta\Omega_i^N \cos(\theta_i^N) \end{Bmatrix}$$

$$\mathbf{E}_{i-1} = \mathbf{\Lambda}_{i-1} \cdot \mathbf{I}_{i-1}$$

Transmittance Becomes a Matrix Expression

$$T^{f}_{2,\{1,2\}}(\theta_2, \phi_2; \theta_0, \phi_0) = \int d\,\Omega_1 \cos(\theta_1)\tau^{f}_{2}(\theta_2, \phi_2; \theta_1, \phi_1)\tau^{f}_{1}(\theta_1, \phi_1; \theta_0, \phi_0)$$

$$\Downarrow$$

$$\mathbf{T}^{f}_{2,\{1,2\}} = \boldsymbol{\tau}^{f}_{2} \cdot \boldsymbol{\Lambda}_1 \cdot \boldsymbol{\tau}^{f}_{1}$$

Matrix Formulation of Nonspecular Transmittance

Outgoing Radiance Distribution

Incident Irradiance Distribution

$$
\begin{Bmatrix} I(\theta^1,\phi^1) \\ I(\theta^2,\phi^2) \\ \dots \\ I(\theta^N,\phi^N) \end{Bmatrix}
=
\begin{Bmatrix} \tau_{11} & \tau_{12} & \dots & \tau_{1N} \\ \tau_{21} & \tau_{22} & \dots & \dots \\ \dots & \dots & \text{etc}\dots & \dots \\ \tau_{N1} & \dots & \dots & \tau_{NN} \end{Bmatrix}
\cdot
\begin{Bmatrix} E(\theta^1,\phi^1) \\ E(\theta^2,\phi^2) \\ \dots \\ E(\theta^N,\phi^N) \end{Bmatrix}
$$

Transmittance Matrix

Elements of the Transmittance Matrix are Biconical Transmittance Distribution Functions

$$\tau_{12} = \frac{\text{Outgoing Radiance } (\theta_1, \phi_1)}{\text{Incoming Irradiance } (\theta_2, \phi_2)}$$

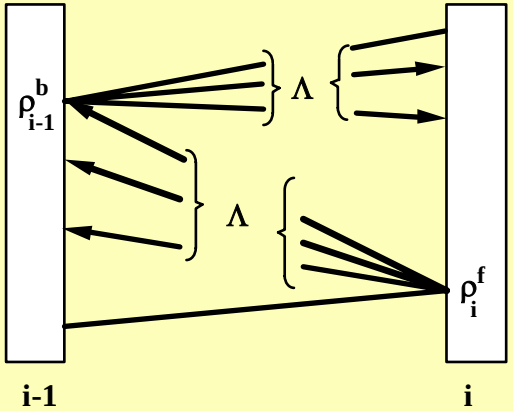
Layer Reflectance Matrix

$$\rho_i^f = \begin{Bmatrix} \rho_i^f(\theta_i^{r,1}, \phi_i^{r,1}; \theta_{i-1}^1, \phi_{i-1}^1) & \rho_i^f(\theta_i^{r,1}, \phi_i^{r,1}; \theta_{i-1}^2, \phi_{i-1}^2) & \dots & \rho_i^f(\theta_i^{r,1}, \phi_i^{r,1}; \theta_{i-1}^N, \phi_{i-1}^N) \\ \rho_i^f(\theta_i^{r,2}, \phi_i^{r,2}; \theta_{i-1}^1, \phi_{i-1}^1) & \rho_i^f(\theta_i^{r,2}, \phi_i^{r,2}; \theta_{i-1}^2, \phi_{i-1}^2) & \dots & \dots \\ \dots & \dots & \dots & \dots \\ \rho_i^f(\theta_i^{r,N}, \phi_i^{r,N}; \theta_{i-1}^1, \phi_{i-1}^1) & \dots & \dots & \rho_i^f(\theta_i^{r,N}, \phi_i^{r,N}; \theta_{i-1}^N, \phi_{i-1}^N) \end{Bmatrix}$$

Backward-going reflected radiance:

$$\mathbf{J}_i = \rho_i^f \cdot \mathbf{E}_{i-1}$$

Interreflections Between Layers



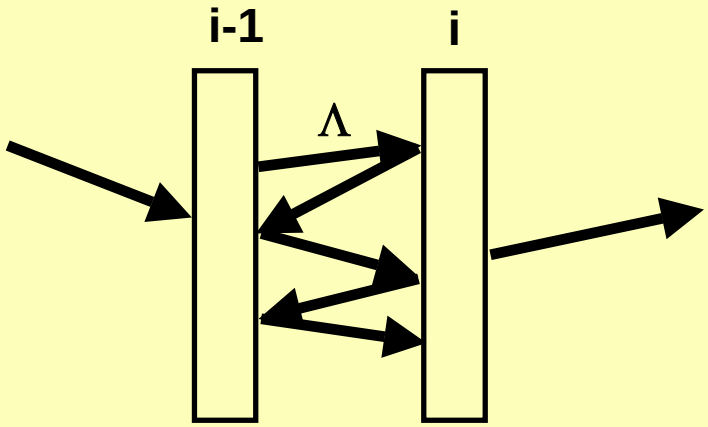
Single-pass interreflection: $\mathbf{M}^{(1)} = \Lambda \cdot \rho^b_{i-1} \cdot \Lambda \cdot \rho^f_i$

Sum of all interreflections: $\mathbf{1} + \sum_{n=1}^{\infty} \mathbf{M}^{(n)} = \left(\mathbf{1} - \Lambda_{i-1}\rho^b_{i-1}\Lambda_i\rho^f_i\right)^{-1}$

Interreflections

- **The sum-of-geometric series argument familiar in dealing with specular multiple reflections also works for the matrix calculation**

2-Layer System Transmittance Matrix



$$\mathbf{T}^{f}_{2\{i-1,i\}} = \tau^{f}_{i} \cdot \left(\mathbf{1} - \Lambda \cdot \rho^{b}_{i-1} \cdot \Lambda \cdot \rho^{f}_{i}\right)^{-\mathbf{1}} \cdot \Lambda \cdot \tau^{f}_{i-1}$$

System Property Notation

capitol T means system transmittance

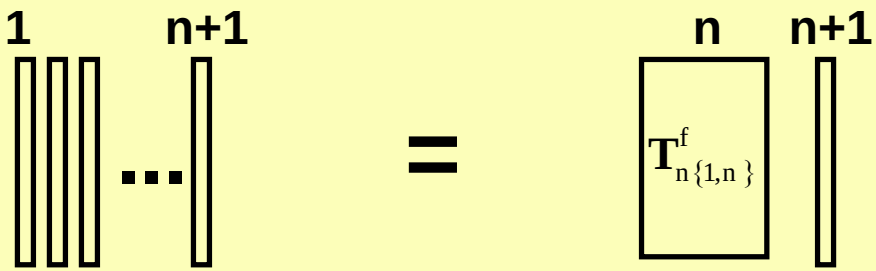
front

$$\mathbf{T}^{f}_{2\{i-1,i\}}$$

a 2-layer (sub) system

(sub)system includes layers i-1 thru i

Many-Layer Systems

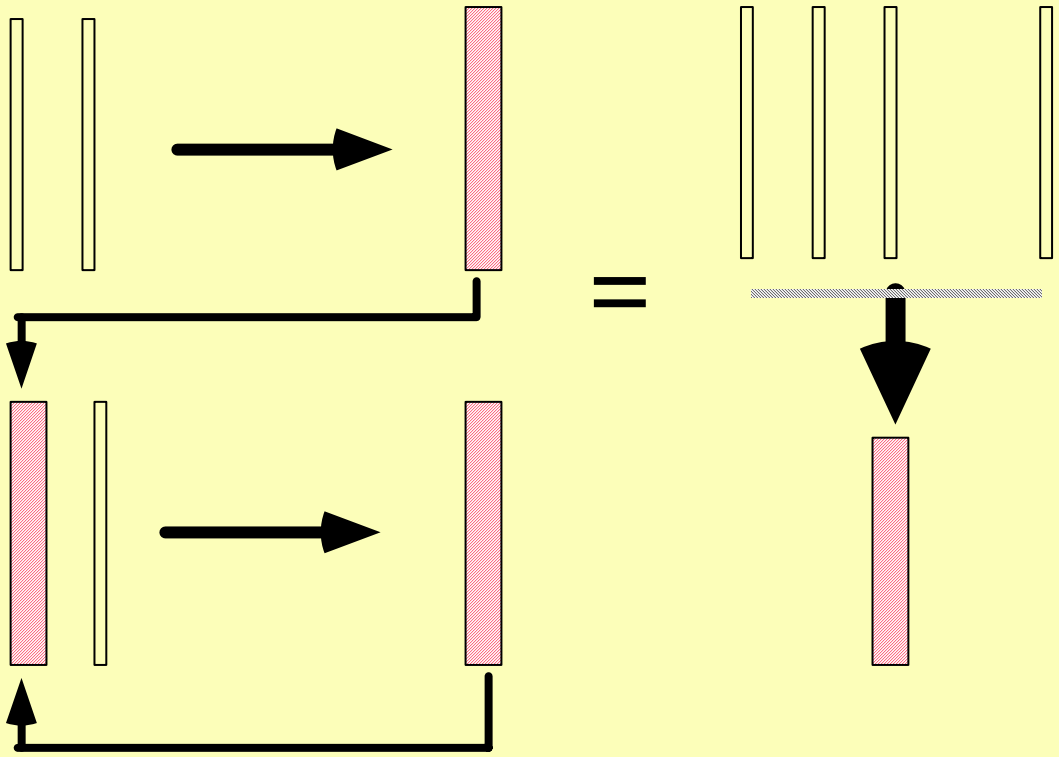


subsystem back reflectance

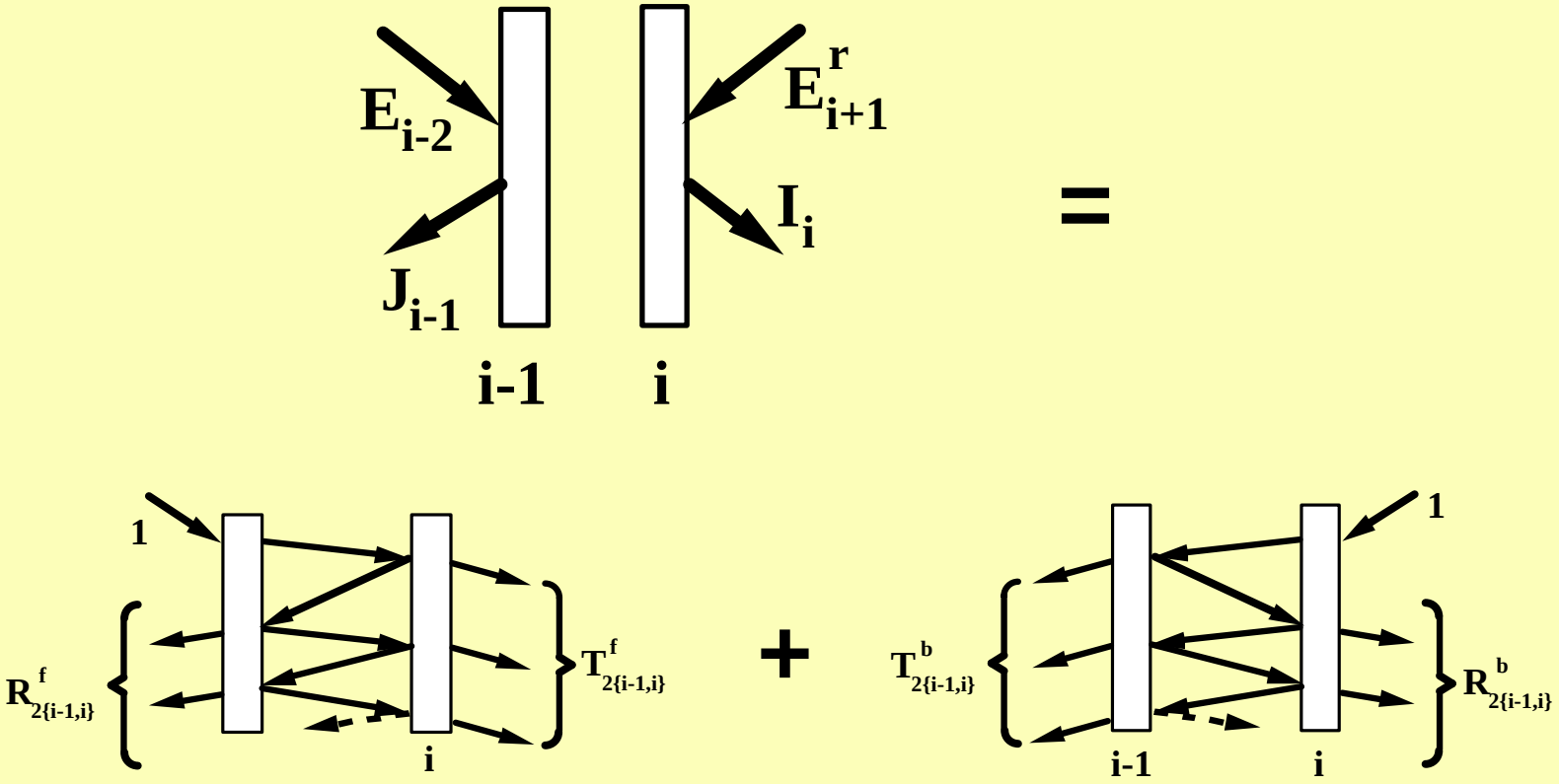
$$\mathbf{T}^{f}_{n+1\{1,n+1\}} = \tau^{f}_{n+1} \cdot \left(\mathbf{1} - \Lambda \cdot \mathbf{R}^{b}_{n\{1,n\}} \cdot \Lambda \cdot \rho^{f}_{n+1}\right)^{-1} \cdot \Lambda \cdot \mathbf{T}^{f}_{n\{1,n\}}$$

a “recursion relation”

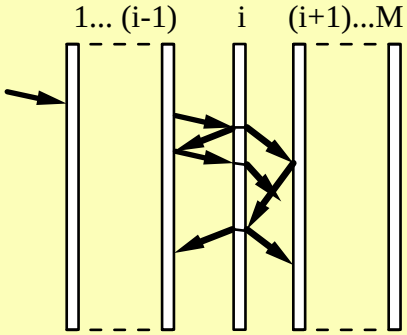
Recursive Calculation of Multilayer System Properties

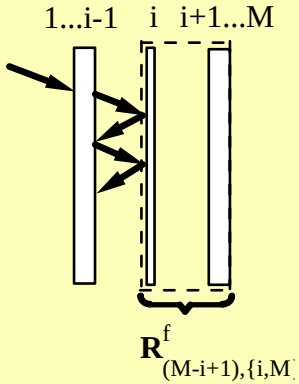


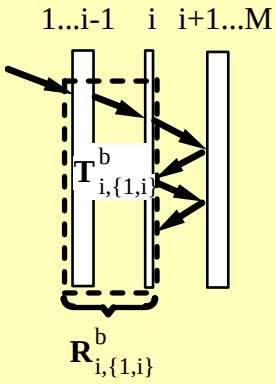
Interior Subsystems



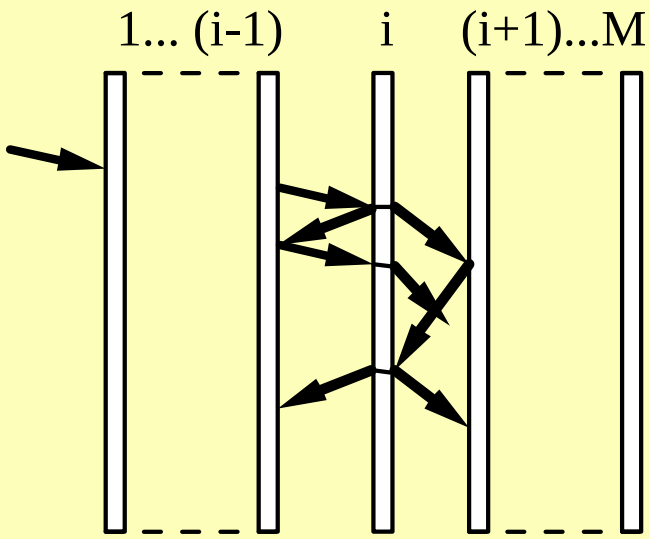
Absorption by an Interior Layer







Absorptance of a Layer in an M-layer System



$$\mathbf{A}^{f}_{i;M} = \alpha^{f}_{i} \cdot (\mathbf{1} - \Lambda \cdot \mathbf{R}^{b}_{i-1,\{1,i-1\}} \cdot \Lambda \cdot \mathbf{R}^{f}_{(M-i+1),\{i,M\}})^{-1} \cdot \Lambda \cdot \mathbf{T}^{f}_{i-1,\{1,i-1\}}$$
$$+\alpha^{b}_{i} \cdot (\mathbf{1} - \Lambda \cdot \mathbf{R}^{f}_{(M-i),\{i+1,M\}} \cdot \Lambda \cdot \mathbf{R}^{b}_{i,\{1,i\}})^{-1} \cdot \Lambda \cdot \mathbf{R}^{f}_{(M-i),\{i+1,M\}} \cdot \Lambda \cdot \mathbf{T}^{f}_{i,\{1,i\}}$$

Hemispherical Transmittance and Layer Absorptance

$$\mathbf{u}^{\mathbf{T}} = \{1 \quad 1 \quad \ldots \quad 1\}$$

$$\boldsymbol{\tau}_i^{f,DH} = \mathbf{u}^{\mathbf{T}} \cdot \Lambda \cdot \boldsymbol{\tau}_i^{f}$$

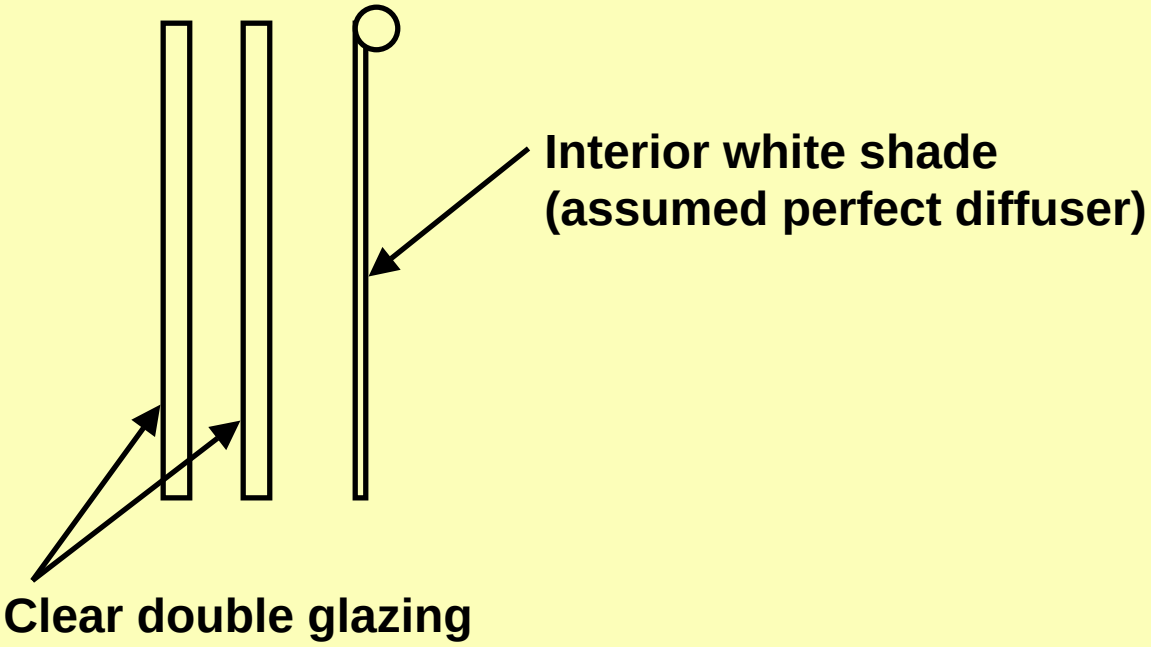
$$\mathbf{T}_M^{f,DH} = \mathbf{u}^{\mathbf{T}} \cdot \Lambda \cdot T_{M,\{1,M\}}^{f}$$

$$\mathbf{T}_M^{f,DH} = \left\{T_{fH}(\theta_0^1, \phi_0^1) \quad T_{fH}(\theta_0^2, \phi_0^2) \quad \ldots \quad T_{fH}(\theta_0^N, \phi_0^N)\right\}$$

$$\mathbf{A}_{i;M}^{f} = \left\{A_{fi}(\theta_0^1, \phi_0^1) \quad A_{fi}(\theta_0^2, \phi_0^2) \quad \ldots \quad A_{fi}(\theta_0^N, \phi_0^N)\right\}$$

The Matrix Calculation Produces All of the Solar-Optical Quantities Necessary, Once the Layer Properties are Known

A Sample Calculation



Network Definition (Azimuthal Symmetry)

$$\theta_l = \{0^o \quad 15^o \quad 30^o \quad 45^o \quad 60^o \quad 75^o \quad 86.25^o\}$$

$$\Lambda = \begin{pmatrix} 0.054 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.407 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.704 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.813 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.704 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.407 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.054 \end{pmatrix}$$

Shade Transmittance Matrix

$$\boldsymbol{\tau}_3^{\mathrm{f}} = \boldsymbol{\tau}_3^{\mathrm{b}} = \begin{pmatrix} 0.067 & 0.067 & 0.067 & 0.067 & 0.067 & 0.067 & 0.067 \\ 0.067 & 0.067 & 0.067 & 0.067 & 0.067 & 0.067 & 0.067 \\ 0.067 & 0.067 & 0.067 & 0.067 & 0.067 & 0.067 & 0.067 \\ 0.067 & 0.067 & 0.067 & 0.067 & 0.067 & 0.067 & 0.067 \\ 0.067 & 0.067 & 0.067 & 0.067 & 0.067 & 0.067 & 0.067 \\ 0.067 & 0.067 & 0.067 & 0.067 & 0.067 & 0.067 & 0.067 \\ 0.067 & 0.067 & 0.067 & 0.067 & 0.067 & 0.067 & 0.067 \end{pmatrix}$$

Shade Reflectance Matrix

$$\rho_3^f = \rho_3^b = \begin{pmatrix} 0.198 & 0.198 & 0.198 & 0.198 & 0.198 & 0.198 & 0.198 \\ 0.198 & 0.198 & 0.198 & 0.198 & 0.198 & 0.198 & 0.198 \\ 0.198 & 0.198 & 0.198 & 0.198 & 0.198 & 0.198 & 0.198 \\ 0.198 & 0.198 & 0.198 & 0.198 & 0.198 & 0.198 & 0.198 \\ 0.198 & 0.198 & 0.198 & 0.198 & 0.198 & 0.198 & 0.198 \\ 0.198 & 0.198 & 0.198 & 0.198 & 0.198 & 0.198 & 0.198 \\ 0.198 & 0.198 & 0.198 & 0.198 & 0.198 & 0.198 & 0.198 \end{pmatrix}$$

Clear Glass Optical Properties

$$\tau_S(\theta_l) = \{0.856 \quad 0.855 \quad 0.850 \quad 0.832 \quad 0.773 \quad 0.532 \quad 0.152\}$$

$$\rho_S(\theta_l) = \{0.077 \quad 0.077 \quad 0.079 \quad 0.093 \quad 0.146 \quad 0.387 \quad 0.818\}$$

Glass Transmittance Matrix

$$\boldsymbol{\tau}_1^{\mathrm{f}} = \boldsymbol{\tau}_1^{\mathrm{b}} = \boldsymbol{\tau}_2^{\mathrm{f}} = \boldsymbol{\tau}_2^{\mathrm{b}} = \begin{pmatrix} 15.99 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2.10 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1.21 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1.02 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1.10 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1.31 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2.84 \end{pmatrix}$$

Glass Reflectance Matrix

$$\rho_1^f = \rho_1^b = \rho_2^f = \rho_2^b = \begin{pmatrix} 1.44 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.189 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.112 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.114 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.207 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.952 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 15.28 \end{pmatrix}$$

Double Glazing Subsystem Matrices

$$\mathbf{T}^{f}_{2\{1,2\}} = \mathbf{T}^{b}_{2\{1,2\}} = \begin{pmatrix} 13.77 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1.81 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1.03 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.859 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.867 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.819 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1.30 \end{pmatrix}$$

$$\mathbf{R}^{f}_{2\{1,2\}} = \mathbf{R}^{b}_{2\{1,2\}} = \begin{pmatrix} 2.50 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.329 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.194 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.194 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.334 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1.27 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 16.35 \end{pmatrix}$$

System Transmittance Matrix

$$\mathbf{T}_3^{\mathrm{f}} = \begin{pmatrix} 0.058 & 0.058 & 0.057 & 0.055 & 0.048 & 0.026 & 0.005 \\ 0.058 & 0.058 & 0.057 & 0.055 & 0.048 & 0.026 & 0.005 \\ 0.058 & 0.058 & 0.057 & 0.055 & 0.048 & 0.026 & 0.005 \\ 0.058 & 0.058 & 0.057 & 0.055 & 0.048 & 0.026 & 0.005 \\ 0.058 & 0.058 & 0.057 & 0.055 & 0.048 & 0.026 & 0.005 \\ 0.058 & 0.058 & 0.057 & 0.055 & 0.048 & 0.026 & 0.005 \\ 0.058 & 0.058 & 0.057 & 0.055 & 0.048 & 0.026 & 0.005 \end{pmatrix}$$

Hemispherical Transmittance and Layer Absorptances

$$\mathbf{T}_3^{f,DH} = \begin{pmatrix} 0.182 & 0.181 & 0.179 & 0.172 & 0.150 & 0.082 & 0.017 \end{pmatrix}$$

$$\mathbf{A}_{1;3}^{f} = \begin{pmatrix} 0.103 & 0.104 & 0.107 & 0.111 & 0.117 & 0.115 & 0.044 \end{pmatrix}$$

$$\mathbf{A}_{2;3}^{f} = \begin{pmatrix} 0.102 & 0.102 & 0.104 & 0.105 & 0.100 & 0.070 & 0.018 \end{pmatrix}$$

$$\mathbf{A}_{3;3}^{f} = \begin{pmatrix} 0.143 & 0.143 & 0.141 & 0.136 & 0.119 & 0.065 & 0.014 \end{pmatrix}$$

Solar Heat Gain Coefficient for Complex Fenestrations

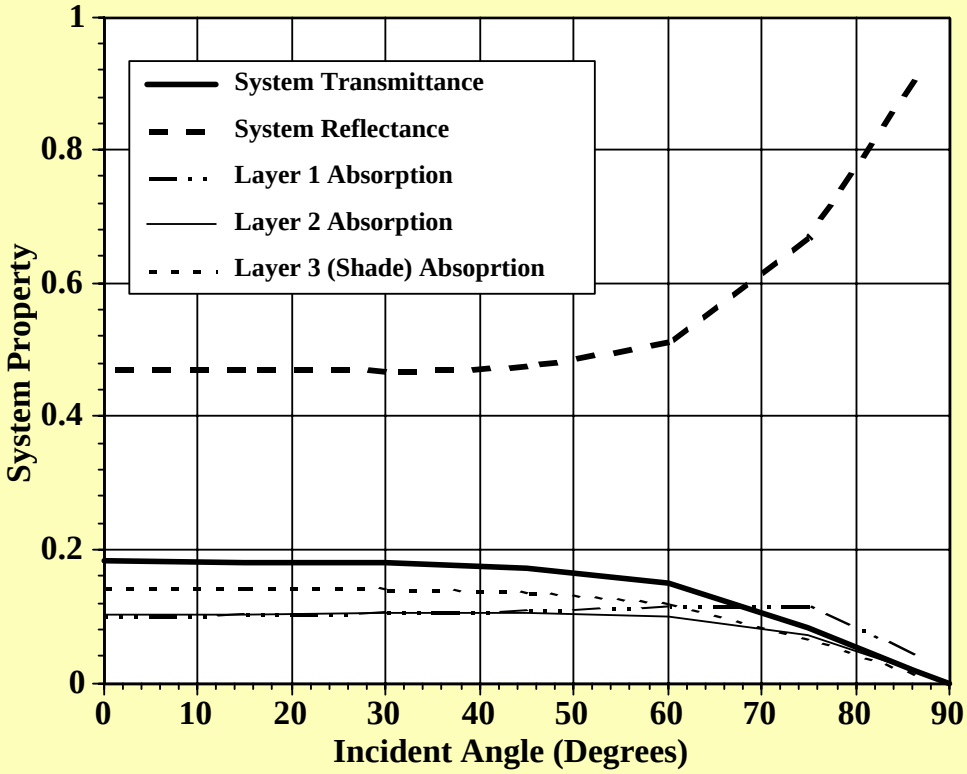
System Directional-Hemispherical Transmittance

$$F(\theta, \phi) = T_{fH}(\theta, \phi) + \sum_{i=1}^{M} N_i A_{fi}(\theta, \phi)$$

Layer Absorptance

Layer Inward-Flowing Fraction

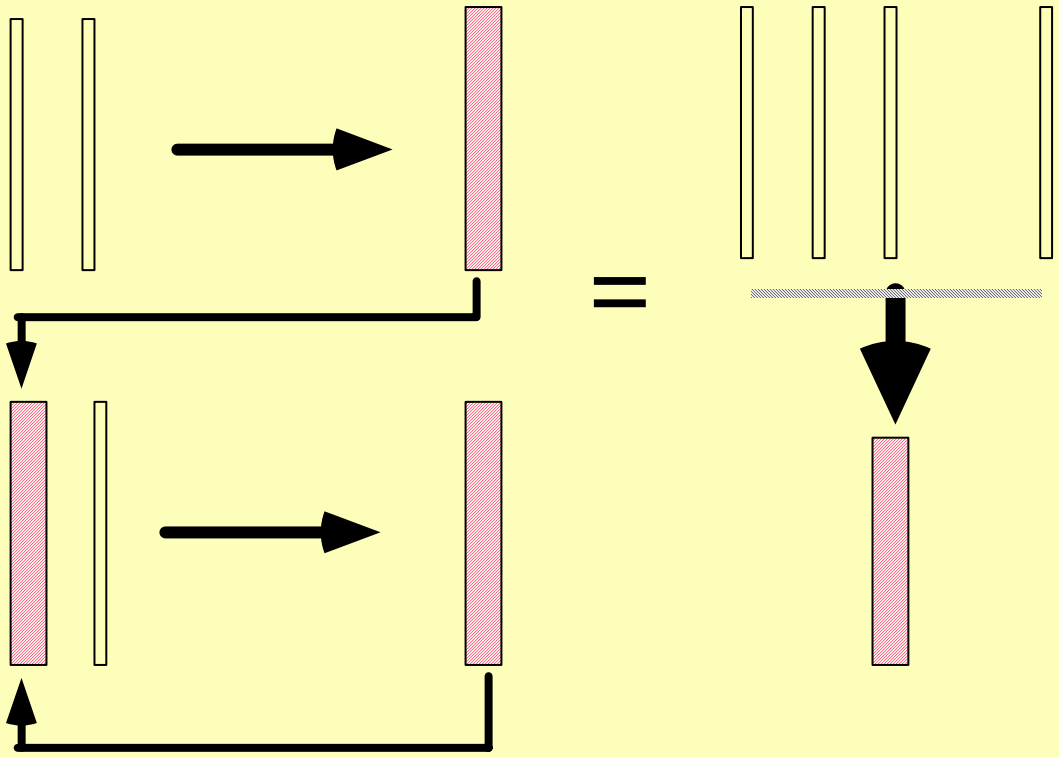
Calculated System Properties Double Glazing with Interior Shade



Conclusion

There is a consistent and systematic way to calculate the solar-optical properties of a fenestration system composed of an arbitrary number of layers, including interreflections, once the bidirectional properties of the layers are known

Recursive Calculation of Multilayer System Properties



Elements of the Transmittance Matrix are Biconical Transmittance Distribution Functions

$$\tau_{12} = \frac{\text{Outgoing Radiance } (\theta_1, \phi_1)}{\text{Incoming Irradiance } (\theta_2, \phi_2)}$$

Matrix Formulation of Nonspecular Transmittance

Outgoing Radiance Distribution

Incident Irradiance Distribution

$$
\begin{Bmatrix} I(\theta^1,\phi^1) \\ I(\theta^2,\phi^2) \\ \dots \\ I(\theta^N,\phi^N) \end{Bmatrix}
=
\begin{Bmatrix} \tau_{11} & \tau_{12} & \dots & \tau_{1N} \\ \tau_{21} & \tau_{22} & \dots & \dots \\ \dots & \dots & \text{etc}\dots & \dots \\ \tau_{N1} & \dots & \dots & \tau_{NN} \end{Bmatrix}
\cdot
\begin{Bmatrix} E(\theta^1,\phi^1) \\ E(\theta^2,\phi^2) \\ \dots \\ E(\theta^N,\phi^N) \end{Bmatrix}
$$

Transmittance Matrix